Long-Time Dynamics of Variable Coefficient mKdV Solitary Waves*

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Abstract

We study the Korteweg-de Vries-type equation $\partial_t u = -\partial_x \left(\partial_x^2 u + f(u) - b(t,x)u\right)$, where b is a small and bounded, slowly varying function and f is a nonlinearity. Many variable coefficient KdV-type equations can be rescaled into this equation. We study the long time behaviour of solutions with initial conditions close to a stable, $b = 0$ solitary wave. We prove that for long time intervals, such solutions have the form of the solitary wave, whose centre and scale evolve according to a certain dynamical law involving the function $b(t,x)$, plus an $H^1(\mathbb{R})$-small fluctuation.

1 Introduction

We study the long time behaviour of solutions to a class Korteweg-de Vries-type equations, with an additional term $b(t,x)u$. These equations, from now on called the bKdV, are of the form

$$\partial_t u = -\partial_x \left(\partial_x^2 u + f(u) - b(t,x)u\right), \tag{1}$$

where $b(t,x)$ is a real valued function and f is a nonlinearity. In this paper we consider a restricted class of nonlinearities. In particular, for monomial nonlinearities, we give a result only for $f(u) = u^3$, corresponding to the modified KdV (mKdV). When $b = 0$, Equation (1) reduces to the generalized Korteweg-de Vries equation (GKdV)

$$\partial_t u = -\partial_x(\partial_x^2 u + f(u)). \tag{2}$$

A remarkable property of the GKdV is the existence of spatially localized solitary (or travelling) waves, i.e. solutions of the form $u = Q_c(x - a - ct)$, where $a \in \mathbb{R}$ and c in some interval I. When $f(u) = u^p$ and $p \geq 2$, solitary waves are explicitly computed to be

$$Q_c(x) = c^{\frac{1}{p-1}} Q(c^{\frac{1}{2}} x),$$

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where

$$Q(x) = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \left(\cosh\left(\frac{p-1}{2}x\right)\right)^2.$$

It is generally believed that an arbitrary, say $H^1(\mathbb{R})$, solution to equation (2) eventually breaks up into a collection of solitary waves and radiation. A discussion of this phenomenon for the generalized KdV appears in Bona [9]. For the general, but integrable case see Deift and Zhou [16].

The mKdV equation is fundamental in many areas of applied mathematics ranging from traffic flow to plasma physics (see [29, 13, 30, 32]) and arises from an approximation of a more complicated systems. The effects of higher order processes can often be collected into a term of the form $b(t,x)u$. Our main result stated at the end of the next section gives, for long time, an explicit, leading order description of a solution to the bKdV initially close to a solitary wave solution of the GKdV.

We assume that the coefficient b and nonlinearity f are such that (1) has global solutions for $H^1(\mathbb{R})$ data and that (1) with $b=0$ possesses solitary wave solutions. Precise conditions will be formulated in the next section. Here we mention that the literature regarding well-posedness of the KdV ($b=0$, $f(u)=u^2$) is extensive and well developed. The Miura transform (see [31]) then gives well-posedness results for the mKdV. Bona and Smith [8] proved global wellposedness of the KdV in $H^2(\mathbb{R})$. See also [25]. Kenig, Ponce, and Vega [27] have proved local wellposedness in $H^s(\mathbb{R})$ for $s \geq -\frac{3}{4}$ and similar results are available for the generalized KdV ($b=0$, monomial nonlinearity $f(u)=u^p$ with $p=2,3,4$)[26]. In particular, local well-posedness for the mKdV in $H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$ and global well-posedness for $s \geq 1$ are known. More recently, results extending local wellposedness in negative index Sobolev spaces to global wellposedness have been proven [15, 14]. There is little literature on global well-posedness of the bKdV in energy space, however, under a smallness assumption on the coefficient b, Dejak and Sigal [17] proved global well-posedness in $H^1(\mathbb{R})$ of the bKdV with $f(u)=u^p$, $p=2,3,4$. They used results of [26], and perturbation and energy arguments.

Soliton solutions of the KdV equation are known to be orbitally stable. Although the linearized analysis of Jeffrey and Kakutani [23] suggested orbital stability, the first nonlinear stability result was given by Benjamin [1]. He assumed smooth solutions and used Lyapunov stability and spectral theory to prove his results. Bona [3] later corrected and improved Benjamin's result to solutions in $H^2(\mathbb{R})$. Weinstein [42] used variation methods, avoiding the use of an explicit spectral respresentaion, and extended the orbital stability result to the GKdV. More recently, Grillakis, Satah, and Strauss [21] extended the Lyapunov method to abstract Hamiltonian systems with symmetry. Numerical simulations of soliton dynamics for the KdV were performed by Bona et al. See [4, 5, 6, 7].

For nonlinear Schrödinger and Hartree equations, long-time dynamics of solitary waves were studied by Bronski and Gerrard [10], Fröhlich, Tsai and Yau [19], Keraani [28], and Fröhlich, Gustafson, Jonsson, and Sigal [18, 24]. For related results and techniques for the nonlinear Schrödinger equations see also [11, 12, 20, 36, 35, 41, 40, 39, 37].

In our approach we use the fact that the bKdV is a (non-autonomous, if b depends on time) Hamiltonian system. As in the case of the nonlinear Schrödinger equation (see [18]), we construct a Hamiltonian reduction of this original, infinite dimensional dynamical system to a two dimensional dynamical system on a manifold of soliton configurations. The analysis of the general bKdV immediately runs into the problem that the natural symplectic form ω is not defined on the tangent space of the soliton manifold. In this paper we prove the main theorem in the cases where the symplectic form is well defined on the tangent space. One such case is when the nonlinearity is $f(u)=u^3$. For the general case see [17]. We remark here that the dynamics for the special case considered here include the higher order correction terms for the scaling parameter c, which

cannot be included in the general case.

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2 Preliminaries, Assumptions, Main Results

The bKdV can be written in Hamiltonian form as

$$\partial_t u = \partial_x H_b'(u), \tag{3}$$

where H_b' is the $L^2(\mathbb{R})$ function corresponding to the Fréchet derivative ∂H_b in the $L^2(\mathbb{R})$ pairing. Here the Hamiltonian H_b is

$$H_b(u) := \int_{-\infty}^{\infty} \frac{1}{2}(\partial_x u)^2 - F(u) + \frac{1}{2}b(t,x)u^2\, dx,$$

where the function F is the antiderivative of f with $F(0) = 0$. The operator ∂_x is the anti-self-adjoint operator (symplectic operator) generating the Poisson bracket

$$\{G_1, G_2\} := \frac{1}{2}\int_{-\infty}^{\infty} G_1'(u)\partial_x G_2'(u) - G_2'(u)\partial_x G_1'(u)\, dx,$$

defined for any G_1, G_2 such that $G_1', G_2' \in H^{\frac{1}{2}}(\mathbb{R})$. The corresponding symplectic form is

$$\omega(v_1,\, v_2) := \frac{1}{2}\int_{-\infty}^{\infty} v_1(x)\partial_x^{-1}v_2(x) - v_2(x)\partial_x^{-1}v_1(x)\, dx,$$

defined for any $v_1, v_2 \in L^1(\mathbb{R})$. Here the operator ∂_x^{-1} is defined as

$$\partial_x^{-1}v(x) := \int_{-\infty}^{x} v(y)\, dy.$$

Note that $\partial_x^{-1} \cdot \partial_x = I$ and, on the space $\{u \in L^2(\mathbb{R}) \,|\, \int_{-\infty}^{\infty} u\, dx = 0\}$, ∂_x^{-1} is formally anti-self-adjoint with inverse ∂_x. Hence, if $\int_{-\infty}^{\infty} v_1(x)\, dx = 0$, then $\omega(v_1,\, v_2) = \int_{-\infty}^{\infty} v_1(x)\partial_x^{-1}v_2(x)\, dx$.

Note that if b depends on time t, then equation (3) is non-autonomous. It is, however, in the form of a conservation law, and hence the integral of the solution u is conserved provided u and its derivatives decay to zero at infinity:

$$\frac{d}{dt}\int_{-\infty}^{\infty} u\, dx = 0.$$

There are also conserved quantities associated to symmetries of (1) when $b = 0$. The simplest such corresponds to time translation invariance and is the Hamiltonian itself. This is also true if b is non-zero but time

independent. If the potential $b = 0$, then (1) is also spatially translation invariant. Noether's theorem then implies that the flow preserves the momentum

$$P(u) := \frac{1}{2}\|u\|_{L^2}^2.$$

In general, when $b \neq 0$ the temporal and spatial translation symmetries are broken, and hence, the Hamiltonian and momentum are no longer conserved. Instead, one has the relations

$$\frac{d}{dt}H_b(u) = \frac{1}{2}\int_{-\infty}^{\infty}(\partial_t b)u^2\,dx, \tag{4}$$

$$\frac{d}{dt}P(u) = \frac{1}{2}\int_{-\infty}^{\infty} b'u^2\,dx, \tag{5}$$

where $b'(t,x) := \partial_x b(t,x)$. For later use, we also state the relation

$$\frac{d}{dt}\frac{1}{2}\int_{-\infty}^{\infty} bu^2\,dx = \int_{-\infty}^{\infty}\frac{1}{2}u^2\partial_t b + b'\left(uf(u) - \frac{3}{2}(\partial_x u)^2 - F(u)\right) - b''u\partial_x u\,dx. \tag{6}$$

Assuming (1) is well-posed in $H^2(\mathbb{R})$, the above equalities are obtained after multiple integration by parts. Then, by density of $H^2(\mathbb{R})$ in $H^1(\mathbb{R})$, the equalities continue to hold for solutions in $H^1(\mathbb{R})$. To avoid these technical details, we assume the Hamiltonian flow on $H^1(\mathbb{R})$ enjoys (4), (5) and (6).

Consider the GKdV, i.e. equation (2). Under certain conditions on f, this equation has travelling wave solutions of the form $Q_c(x-ct)$, where Q_c a positive $H^2(\mathbb{R})$ function. Substituting $u = Q_c(x-ct)$ into the GKdV gives the scalar field equation

$$-\partial_x^2 Q_c + cQ_c - f(Q_c) = 0. \tag{7}$$

Existence of solutions to this equation has been studied by numerous authors. See [38, 2]. In particular, Berestyki and Lions [2] give sufficient and necessary conditions for a positive and smooth solution Q_c to exist. We assume $g := -cu + f(u)$ satisfies the following conditions:

1. g is locally Lipschitz and $g(0) = 0$,
2. $x^* := \inf\{x > 0 \,|\, \int_0^x g(y)\,dy\}$ exists with $x^* > 0$ and $g(x^*) > 0$, and
3. $\lim_{s\to 0}\frac{g(s)}{s} \leq -m < 0$.

Then, as shown by Berestycki and Lions, (7) has a unique (modulo translations) solution $Q_c \in C^2$ for c in some interval, which is positive, even (when centred at the origin), and with Q_c, $\partial_x Q_c$, and $\partial_x^2 Q_c$ exponentially decaying to zero at infinity ($\partial_x Q_c < 0$ for $x > 0$). Furthermore, if f is C^2, then the implicit function theorem implies that Q_c is C^2 with respect to the parameter c on some interval $I_0 \subset \mathbb{R}_+$. We assume that $x^m\partial_c^n Q_c \in L^1(\mathbb{R})$ for $n = 1,2$, $m = 0,1,2$ so that integrals containing $\partial_c^n Q_c$ are continuous and differentiable with respect to c. We also make the assumption that

$$\int_{-\infty}^{\infty}\partial_c Q_c\,dx = 0 \tag{8}$$

for all $c \in I$. This implies that

$$\int_{-\infty}^{x}\partial_c Q_c(z)\,dz, \int_{-\infty}^{x}\partial_c^2 Q_c(z)\,dz \in L^2(\mathbb{R}). \tag{9}$$

To see this use the isometry property of the Fourier transform and the decay properties of $\partial_c Q_c$. The above requirements of Q_c are implicit assumptions on the nonlinearity f and are true when $f(u) = u^3$. Assumption (8) is a very important and restrictive requirement; it does not hold when $f(x) = x^p$ and $p \neq 3$. For the case where (8) does not hold see [17].

The solitary waves Q_c are orbitally stable if $\delta'(c) > 0$, where $\delta(c) = P(Q_c)$. See Weinstein [42] the first proof for general nonlinearities. Moreover, in [21], Grillakis, Shatah and Strauss proved that $\delta'(c) > 0$ is a necessary and sufficient condition for Q_c to be orbitally stable. In this paper, we assume that Q_c is stable for all c in some compact interval $I \subset I_0$, or equivalently that $\delta'(c) > 0$ on I. For $f(u) = u^p$, we have $\delta'(c) = \frac{5-p}{4(p-1)} \|Q_{c=1}\|^2_{L^2}$, which implies the well known stability criterion $p < 5$ corresponding to subcritical power nonlinearities.

The scalar field equation (7) for the solitary wave can be viewed as an Euler-Lagrange equation for the extremals of the Hamiltonian $H_{b=0}$ subject to constant momentum $P(u)$. Moreover, Q_c is a stable solitary wave if and only if it is a minimizer of $H_{b=0}$ subject to constant momentum P. Thus, if c is the Lagrange multiplier associated to the momentum constraint, then Q_c is an extremal of

$$\begin{aligned} \Lambda_{ca}(u) &:= H_{b=0}(u) + cP(u) \\ &= \int_{-\infty}^{\infty} \frac{1}{2}(\partial_x u)^2 + \frac{1}{2}cu^2 - F(u)\,dx, \end{aligned} \tag{10}$$

and hence $\Lambda'_{ca}(Q_c) = 0$.

The functional Λ_{ca} is translationally invariant. Therefore, $Q_{ca}(x) := Q_c(x-a)$ is also an extremal of Λ_{ca}, and $Q_c(x - ct - a)$ is a solitary wave solution of (1) with $b = 0$. All such solutions form the two dimensional C^∞ manifold of solitary waves

$$M_s := \{Q_{ca} \,|\, c \in I, a \in \mathbb{R}\},$$

with tangent space $T_{Q_{ca}} M_s$ spanned by the vectors

$$\zeta^{tr}_{ca} := \partial_a Q_{ca} = -\partial_x Q_{ca} \text{ and } \zeta^{n}_{ca} := \partial_c Q_{ca}, \tag{11}$$

which we call the translation and normalization vectors. Notice that the two tangent vectors are orthogonal in $L^2(\mathbb{R})$.

In addition to the requirement on b that (1) is globally wellposed, we assume the potential b is bounded, twice differentiable, and small in the sense that

$$|\partial_t^n \partial_x^m b| \leq \epsilon_a \epsilon_t^n \epsilon_x^m, \tag{12}$$

for $n = 0, 1$, $m = 0, 1, 2$, and $n + m \leq 2$. The positive constants ϵ_a, ϵ_x, and ϵ_t are amplitude, length, and time scales of the function b. We assume all are less than or equal to one.

Lastly, we make some explicit assumptions on the local nonlinearity f. We require the nonlinearity to be k times continuously differentiable, with $f^{(k)}$ bounded for some $k \geq 3$ and $f(0) = f'(0) = 0$. These assumptions ensure the Hamiltonian is finite on the space $H^1(\mathbb{R})$ and, since Q_c decays exponentially (see [2]), both $f(Q_c)$ and $f'(Q_c)$ have exponential decay.

We are ready to state our main result. Recall that $I_0 \subset \mathbb{R}_+$ is an interval where Q_c is twice continuously differentiable.

Theorem 1. *Let the above assumptions hold and assume $\delta'(c) > 0$ for all c in a compact set $I \subset I_0$. Assume $\epsilon_a \leq 1$. Then, if $\epsilon_x \leq 1$, ϵ_0 and ϵ_t are small enough, there is a positive constant C such that the solution to (1) with an initial condition u_0 satisfying $\inf_{Q_{ca}\in M_s} \|u_0 - Q_{ca}\|_{H^1} \leq \epsilon_0$ can be written as*

$$u(x,t) = Q_{c(t)}(x - a(t)) + \xi(x,t),$$

where $\|\xi(t)\|_{H^1} = \mathrm{O}\left(\epsilon_0 + (\epsilon_a\epsilon_x\epsilon_0)^{\frac{1}{2}} + \epsilon_x + \epsilon_t\right)$ for all times $t \leq C(\epsilon_a\epsilon_x)^{-1}$. Moreover, during this time interval the parameters $a(t)$ and $c(t)$ satisfy the equations

$$\begin{pmatrix} \dot{a} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} c - b(a) \\ 0 \end{pmatrix} + b'(a)\frac{\delta(c)}{\delta'(c)}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathrm{O}\left((\epsilon_0 + \epsilon_x + \epsilon_t)^2 + (\epsilon_a\epsilon_x\epsilon_0)^{\frac{1}{2}}(\epsilon_x + \epsilon_t + \epsilon_0)\right),$$

where c is assumed to lie in the compact set I.

Sketch of Proof and Paper Organization. To realize the Hamiltonian reduction we decompose functions in a neighbourhood of the soliton manifold M_s as

$$u = Q_{ca} + \xi$$

with ξ symplectically orthogonal to $T_{Q_{ca}}M_s$, i.e. $\xi \perp \partial_x^{-1} T_{Q_{ca}}M_s$. We show that there is an $\epsilon_0 > 0$ such that if the solution u satisifes the estimate $\inf_{Q_{ca}} \|u - Q_{ca}\|_{H^1} < \epsilon_0$, then there are unique C^1 functions $a(u)$ and $c(u)$ such that $u = Q_{c(u)a(u)} + \xi$ with $\xi \perp \partial_x^{-1} T_{Q_{ca}}M_s$.

With the knowledge that the symplectic decomposition exists, we substitute $u = Q_{ca} + \xi$ into the bKdV (1) and split the resulting equation according to the decomposition

$$L^2(\mathbb{R}) = \partial_x^{-1} T_{Q_{ca}}M_s \oplus \left(\partial_x^{-1} T_{Q_{ca}}M_s\right)^{\perp}$$

to obtain equations for the parameters c and a, and an equation for the (infinite dimensional) fluctuation ξ. In Section 4 we isolate the leading order terms in the equations for a and c and estimate the remainder, including all terms containing ξ. In Sections 6 and 7, we establish spectral properties and a lower bound of the Hessian Λ''_{ca} on the space $\left(\partial_x^{-1} T_{Q_{ca}}M_s\right)^{\perp}$.

The proof that $\|\xi\|_{H^1}$ is sufficiently small is the final ingredient in the proof of the main theorem. The remaining sections concentrate on proving this crucial result. We employ a Lyapunov method and in Section 5 we construct the Lyapunov function Γ_c and prove an estimate on its time derivative. This estimate is later time maximized over an interval $[0, T]$, and integrated to obtain an upper bound on Γ_c involving the time T and the norms of ξ. We combine this upper bound with the lower bound on Γ_c following from the results of Section 7, and obtain an inequality involving $\|\xi\|_{H^1}$. In Section 8 we solve the inequality to find an upper bound on $\|\xi\|_{H^1}$ provided $\|\xi(0)\|_{H^1}$ is small enough. We substitute this bound into the bound appearing in the dynamical equation for a and c, and take $\epsilon_a\epsilon_x$ and ϵ_0 small enough so that all intermediate results hold to complete the proof. $\square$

3 Modulation of Solutions

As stated in the previous section, we begin the proof by decomposing the solution of (1) into a modulated solitary wave and a fluctuation ξ:

$$u(x,t) = Q_{c(t)a(t)}(x) + \xi(x,t), \tag{13}$$

with a, c, and ξ fixed by the orthogonality condition

$$\xi \perp \partial_x^{-1} T_{Q_{ca}} M_s, \tag{14}$$

where

$$\partial_x^{-1} : g \mapsto \int_{-\infty}^{x} g(z)\, dz.$$

Note that $\partial_x^{-1} T_{Q_{ca}} M_s$ is a subset of $L^2(\mathbb{R})$ (see (9)).

The existence and uniqueness of parameters a and c such that $\xi = u - Q_{ca}$ satisfies (14) follows from the next lemma concerning a restriction of ∂_x^{-1} and the implicit function theorem.

The restriction K of ∂_x^{-1} to the tangent space $T_{Q_{ca}} M_s$ is defined by the equation $KP_T = P_T \partial_x^{-1} P_T$, where P_T is the orthogonal projection onto $T_{Q_{ca}} M_s$. In the natural basis $\{\zeta_{ca}^{tr}, \zeta_{ca}^{n}\}$ of the tangent space $T_{Q_{ca}} M_s$, the matrix representation of K is $N^{-1}\Omega_{ca}$, where

$$N := \begin{pmatrix} \|\zeta_{ca}^{tr}\|_{L^2}^2 & 0 \\ 0 & \|\zeta_{ca}^{n}\|_{L^2}^2 \end{pmatrix}$$

and

$$\Omega_{ca} := \begin{pmatrix} \langle \zeta_{ca}^{tr}, \partial_x^{-1} \zeta_{ca}^{tr} \rangle & \langle \zeta_{ca}^{n}, \partial_x^{-1} \zeta_{ca}^{tr} \rangle \\ \langle \zeta_{ca}^{tr}, \partial_x^{-1} \zeta_{ca}^{n} \rangle & \langle \zeta_{ca}^{n}, \partial_x^{-1} \zeta_{ca}^{n} \rangle \end{pmatrix}. \tag{15}$$

Recall that $\delta(c) = \frac{1}{2}\|Q_c\|_{L^2}^2$.

Lemma 1. *If $\delta'(c) > 0$ on the compact set $I \subset \mathbb{R}_+$, then the matrix Ω_{ca} is invertible for all $c \in I$, and*

$$\Omega_{ca}^{-1} = \frac{1}{\delta'(c)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{16}$$

Clearly, $\|\Omega_{ca}^{-1}\| \leq \lfloor \delta' \rfloor^{-1}$, where $\lfloor \delta' \rfloor := \inf_I \delta'(c)$.

Proof. The lemma follows from the relations $\langle \zeta_{ca}^{tr}, \partial_x^{-1} \zeta_{ca}^{tr} \rangle = 0$, $\langle \zeta_{ca}^{n}, \partial_x^{-1} \zeta_{ca}^{n} \rangle = 0$ and $\langle \zeta_{ca}^{tr}, \partial_x^{-1} \zeta_{ca}^{n} \rangle = \langle \zeta_{ca}^{n}, Q_c \rangle = \delta'(c)$. □

Given $\varepsilon > 0$, define the tubular neighbourhood $U_\varepsilon := \{u \in L^2(\mathbb{R}) \mid \inf_{(c,\, a) \in I \times \mathbb{R}} \|u - Q_{ca}\|_{L^2} < \varepsilon\}$ of the solitary wave manifold M_s in $L^2(\mathbb{R})$.

Proposition 1. *Let $I \subset \mathbb{R}_+$ be a compact interval such that $c \mapsto Q_{ca}$ is $C^1(I)$. Then there exists a positive number $\varepsilon_0 = \varepsilon_0(I) = \mathrm{O}\left(\lfloor \delta' \rfloor^2\right)$ dependent on I and unique C^1 functions $a : U_{\varepsilon_0} \to \mathbb{R}_+$ and $c : U_{\varepsilon_0} \to I$, such that*

$$\langle Q_{c(u)a(u)} - u, \partial_x^{-1} \zeta_{c(u)a(u)}^{tr} \rangle = 0 \text{ and } \langle Q_{c(u)a(u)} - u, \partial_x^{-1} \zeta_{c(u)a(u)}^{n} \rangle = 0$$

for all $u \in U_{\varepsilon_0}$. Moreover, there is a positive real number $C = C(I)$ such that

$$\|u - Q_{c(u)a(u)}\|_{H^1} \leq C \inf_{Q_{ca} \in M_s} \|u - Q_{ca}\|_{H^1} \tag{17}$$

for all $u \in U_{\varepsilon_0} \cap H^1(\mathbb{R})$.

Proof. Let $\mu := (\mu^1, \mu^2)^T \in \mathbb{R}_+ \times I$ and define $G : \mathbb{R}_+ \times I \times H^1(\mathbb{R}) \to \mathbb{R}^2$ as

$$G : (\mu, u) \mapsto \begin{pmatrix} \langle Q_{ca} - u, \Omega_{ca}\zeta_{ca}^{tr}\rangle \\ \langle Q_{ca} - u, \Omega_{ca}\zeta_{ca}^{n}\rangle \end{pmatrix},$$

where $a = \mu^1$ and $c = \mu^2$. The proposition is equivalent to solving $G(g(u), u) = 0$ for a C^1 function g. Let $\mu_0 = (a\,c)^T$. If G is C^1, $G(\mu_0, Q_{ca})$=0, and $\partial_\mu F(\mu_0, Q_{ca})$ is invertible, then the implicit function theorem asserts the existence of an open ball $B_{\varepsilon_0}(Q_{ca})$ of radius ε_0 with centre Q_{ca}, and a unique function $g_{Q_{ca}} : B_\delta(Q_{ca}) \to \mathbb{R}_+ \times I$, such that $G(g_{Q_{ca}}(u), u) = 0$ for all $u \in B_{\varepsilon_0}(Q_{ca})$. The first two conditions are trivial, and the third follows from Lemma 1 since $\partial_\mu G(\mu_0, Q_{ca}) = \Omega_{ca}$. The radius of the balls $B_\varepsilon(Q_{ca})$ depend on the parameters c and a. To obtain an estimate of the radius, and to show that we can take ε independent of the parameters c and a, we give a proof of the existence of the above function $g_{Q_{ca}}$ for our special case using the contraction mapping principle.

We wish to solve $G(\mu, u) = 0$ for $\mu := (\mu^1, \mu^2)^T$ with u close to Q_{ca} in $L^2(\mathbb{R})$. Expand $G(\mu, u)$ in μ about $\mu_0 = (a\,c)^T$: $G(\mu, u) = G(\mu_0, u) + \partial_\mu G(\mu_0, u)(\mu - \mu_0) + R(\mu, u)$, with $R(\mu, u) = \mathrm{O}\left(\|\mu - \mu_0\|^2\right)$ (G is C^2). Thus, we must solve $\mu = \mu_0 - [\partial_\mu G(\mu_0, u)]^{-1} (G(\mu_0, u) + R(\mu, u))$ for μ. Clearly, since $\partial_\mu G(\mu_0, u) = \Omega_{ca}$, μ must be a fixed point of

$$H_{u\mu_0}(\mu) := \mu_0 - \Omega_{ca}^{-1}[G(\mu_0, u) + R(\mu, u)].$$

We now show that $H_{u\mu_0}$ is a strict contraction, and hence has a fixed point. By the mean value theorem

$$\|H_{u\mu_0}(\mu_2) - H_{u\mu_0}(\mu_1)\| \leq \sup \|\partial_\mu H_{u\mu_0}\| \|\mu_2 - \mu_1\|,$$

where the supremum is taken over all allowed parameter values. Furthermore, we have

$$\begin{aligned}\partial_\mu H_{u\mu_0}(\mu) &= -\Omega_{ca}^{-1}[\partial_\mu G(\mu, u) - \partial_u G(\mu_0, u)] \\ &= -\Omega_{ca}^{-1}[\partial_\mu G(\mu, u) - \partial_\mu G(\mu, Q_{ca}) + \partial_\mu G(\mu, Q_{ca}) - \partial_\mu G(\mu_0, Q_{ca}) + \partial_\mu G(\mu_0, Q_{ca}) - \partial_u G(\mu_0, u)]\end{aligned}$$

Using the mean value theorem again, we compute that

$$\|\partial_\mu G(\mu, u) - \partial_\mu G(\mu_0, u)\| \leq C_1\delta + C_2\varepsilon$$

for some constants C_1 and C_2 if $\|\mu - \mu_0\| < \delta$ and $\|u - Q_{ca}\|_{L^2} < \varepsilon$. Combining all the estimates gives

$$\|H_{u\mu_0}(\mu_2) - H_{u\mu_0}(\mu_1)\| \leq \sup \|\Omega_{ca}^{-1}\| \left(C_1\delta + C_2\varepsilon\right) \|\mu_2 - \mu_1\|.$$

Thus, if $\delta = \frac{1}{4}(C_1 \sup \|\Omega_{ca}^{-1}\|)^{-1}$ and $\varepsilon = \frac{1}{4}(C_2 \sup \|\Omega_{ca}^{-1}\|)^{-1}$, then $H_{u\mu_0}$ is a contraction.

We now choose δ and ε so that $H_{u\mu_0}$ maps $B_\delta(\mu_0)$ to $B_\delta(\mu_0)$. We have that

$$\|H_{u\mu_0} - \mu_0\| \leq \|\Omega_{ca}^{-1}\left(G(\mu_0, u) + R(\mu, u)\right)\| \leq \sup \|\Omega_{ca}^{-1}\| \left(\|G(\mu_0, u) - G(\mu_0, Q_{ca})\| + \mathrm{O}\left(\delta^2\right)\right).$$

By the mean value theorem $\|G(\mu_0, u) - G(\mu_0, Q_{ca})\| \leq C_3\varepsilon$. Thus, if we take $\delta = \mathrm{O}\left(\sup \|\Omega_{ca}^{-1}\|^{-1}\right)$ so that $\mathrm{O}\left(\delta^2\right) \leq \frac{1}{4}\left(\sup \|\Omega_{ca}^{-1}\|\right)^{-1} \delta$, then

$$\|H_{u\mu_0} - \mu_0\| \leq C_3 \sup \|\Omega_{ca}^{-1}\| \varepsilon + \frac{1}{4}\delta.$$

We now take $\varepsilon < \frac{1}{4}\left(C_3 \sup \|\Omega_{ca}^{-1}\|\right)^{-1}\delta$ to obtain $\|H_{u\mu_0} - \mu_0\| \leq \frac{1}{2}\delta$. To complete the argument, take δ to be the smaller of $\frac{1}{4}\left(C_1 \sup \|\Omega_{ca}^{-1}\|\right)^{-1}$ and the above choice, and then ε to be the smaller of $\frac{1}{4}(C_2 \sup \|\Omega_{ca}^{-1}\|)^{-1}$ and $\delta(4C_3 \sup \|\Omega_{ca}^{-1}\|)^{-1}$. Using the bound on $\|\Omega_{ca}^{-1}\|$ we find that

$$\varepsilon = \mathrm{O}\left(\lfloor \delta' \rfloor^2\right)$$

if $\sup \|\Omega_{ca}^{-1}\| \geq 1$, or equivalently, when $\lfloor \delta' \rfloor$ is sufficiently small.

The above argument shows that there exists balls $\{B_\varepsilon(Q_{ca}) \,|\, a \in \mathbb{R}_+, c \in I\}$ with radius ε dependent only on the compact set I. Then, defining $U_{\varepsilon_0} = \bigcup\{B_{\varepsilon_0}(Q_{ca}) \,|\, a \in \mathbb{R}_+, c \in I\}$ and pasting the C^1 functions $g_{Q_{ca}}$ together, into a C^1 function $g_I : U_{\varepsilon_0} \to \mathbb{R}_+ \times I$, proves existence of the required C^1 functions $a(u)$ and $c(u)$. Uniqueness follows from the uniqueness of the functions $g_{Q_{ca}}$.

Let $u \in U_\varepsilon$, $c \in I$, and $a \in \mathbb{R}$, and consider the equation

$$u - Q_{c(u)a(u)} = u - Q_{ca} + Q_{ca} - Q_{c(u)a(u)}.$$

Clearly, inequality (17) will follow if $\|Q_{ca} - Q_{c(u)a(u)}\|_{H^1} \leq C\|u - Q_{ca}\|_{H^1}$ for some positive constant C. Since the derivatives $\partial_c Q_{ca}$ and $\partial_a Q_{ca}$ are uniformly bounded in $H^1(\mathbb{R})$ over $I \times \mathbb{R}$, the mean value theorem gives that $\|Q_{ca} - Q_{c(u)a(u)}\|_{H^1} \leq C\|(c,a)^T - (c(u), a(u))^T\|$, where the constant C does not depend on c, a. The relations $g_I(Q_{ca}) = (c,a)^T$ and $g_I(u) = (c(u), a(u))^T$ then imply $\|Q_{ca} - Q_{c(u)a(u)}\|_{H^1} \leq C\|g_I(Q_{ca}) - g_I(u)\|$. Again, we appeal to the mean value theorem and use the properties of Ω_{ca} and that $\partial_u g_I = \partial_\mu G^{-1} \partial_u G$ is uniformly bounded in the parameters c and a to obtain (17). □

4 Evolution Equations for Parameters ξ, a and c

In Section 3 we proved that if u remains close enough to the solitary wave manifold M_s, then we can write a solution u to (1) uniquely as a sum of a modulated solitary wave Q_{ca} and a fluctuation ξ satisfying the orthogonality condition (14). Thus, as u evolves according to the initial value problem (1), the parameters $a(t)$ and $c(t)$ trace out a path in $\mathbb{R}^2$. The goal of this section is to derive the dynamical equations for the parameters a and c, and the fluctuation ξ. We obtain such equations by substituting the decomposition $u = Q_{ca} + \xi$ into (1) and then projecting the resulting equation onto appropriate directions, with the intent of using the orthogonality condition on ξ.

From now on, u is the solution of (1) with initial condition u_0 satisfying $\epsilon_0 := \inf_{Q_{ca} \in M_s} \|u_0 - Q_{ca}\|_{H^1} < \varepsilon_0$, and $T_0 = T_0(u_0)$ is the maximal time such that $u(t) \in U_\varepsilon$ for $0 \leq t \leq T_0$. Then for $0 \leq t \leq T_0$, u can be decomposed as in (13) and (14).

Proposition 2. *Assume $\delta'(c) \neq 0$. Say $u = Q_{ca} + \xi$ is a solution to (1), where ξ satisfies (14). Then, if $\|\xi\|_{H^1}$ is small enough, $\epsilon_x \leq 1$, and $c \in I$,*

$$\begin{pmatrix} \dot{a} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} c - b(t,a) \\ 0 \end{pmatrix} + b'(t,a)\frac{\delta(c)}{\delta'(c)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + Z(a, c, \xi), \tag{18}$$

where $Z(a, c, \xi) = \mathrm{O}\left(\epsilon_a \epsilon_x^2 + \epsilon_a \epsilon_x \|\xi\|_{H^1} + \|\xi\|_{H^1}^2\right)$.

Proof. Recall that the solitary wave Q_{ca} is an extremal of the functional Λ_{ca}. To use this fact we rearrange definition (10) of Λ_{ca} to write the Hamiltonian H_b as

$$H_b(u) = \Lambda_{ca}(u) - cP(u) + \frac{1}{2}\int_{-\infty}^{\infty} bu^2(x)\,dx,$$

where for notational simplicity we have suppressed the space and time dependency of b. Substituting $Q_{ca}+\xi$ for u in (3) and using the above expression for H_b gives the equation

$$\dot{a}\zeta_{ca}^{tr} + \dot{c}\zeta_{ca}^{n} + \dot{\xi} = \partial_x \Lambda'_{ca}(Q_{ca}+\xi) - c\partial_x[Q_{ca}+\xi] + \partial_x[(Q_{ca}+\xi)b],$$

where dots indicate time differentiation. Let $\mathcal{L}_Q := \Lambda''_{ca}(Q_{ca})$,

$$\delta b := b(t,x) - b(t,a)$$

and

$$\delta^2 b := b(t,x) - b(t,a) - b'(t,a)(x-a).$$

Taylor expanding $\Lambda'_{ca}(Q_{ca}+\xi)$ to linear order in ξ, using that Q_{ca} is an extremal of Λ_{ca} and the relation $\zeta_{ca}^{tr} = -\partial_x Q_{ca}$ gives that

$$\begin{aligned}\dot{\xi} = \partial_x\left[(\mathcal{L}_Q + \delta b + b(a) - c)\xi\right] + \partial_x N'_{ca}(\xi) - [\dot{a} - c + b(a)]\zeta_{ca}^{tr} - \dot{c}\zeta_{ca}^{n}\\ + b'(a)\partial_x[(x-a)Q_{ca}] + \partial_x[\delta^2 b Q_{ca}].\end{aligned} \tag{19}$$

The nonlinear terms have been collected into $N'_{ca}(\xi)$ given by (28) in Appendix A.

Define the vectors $\zeta_1 := \zeta_{ca}^{tr}$ and $\zeta_2 := \zeta_{ca}^{n}$. Projecting (19) onto $\partial_x^{-1}\zeta_i$ for $i = 1, 2$ and using the antisymmetry of ∂_x gives the two equations

$$\begin{aligned}[\dot{a} - c + b(a)]\left[\langle\zeta_{ca}^{tr}, \partial_x^{-1}\zeta_i\rangle + \langle\xi,\zeta_i\rangle\right] + \dot{c}\langle\zeta_{ca}^{n}, \partial_x^{-1}\zeta_i\rangle + \langle\dot{\xi}, \partial_x^{-1}\zeta_i\rangle - \dot{a}\langle\xi,\zeta_i\rangle = -b'(t,a)\langle(x-a)Q_{ca},\zeta_i\rangle\\ - \langle\delta^2 b Q_{ca},\zeta_i\rangle - \langle\delta b\xi,\zeta_i\rangle - \langle N'_{ca}(\xi),\zeta_i\rangle - \langle\mathcal{L}_Q\xi,\zeta_i\rangle.\end{aligned} \tag{20}$$

We can replace the term containing $\dot{\xi}$ since the time derivative of the orthogonality condition $\langle\xi, \partial_x^{-1}\zeta_i\rangle = 0$ implies $\langle\dot{\xi}, \partial_x^{-1}\zeta_i\rangle = \dot{a}\langle\xi,\zeta_i\rangle - \dot{c}\langle\xi, \partial_c\partial_x^{-1}\zeta_i\rangle$. Note that we have used the relation $\partial_a\zeta_i = -\partial_x\zeta_i$. Thus, in matrix form, (20) becomes

$$(I+B)\Omega_{ca}\begin{pmatrix}\dot{a} - c + b(t,a)\\ \dot{c}\end{pmatrix} = X + Y, \tag{21}$$

where

$$X := -b'(t,a)\delta(c)\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}\langle\delta^2 b Q_{ca},\zeta_{ca}^{tr}\rangle\\ \langle\delta^2 b Q_{ca},\zeta_{ca}^{n}\rangle\end{pmatrix},$$

$$Y := -\begin{pmatrix}\langle\delta b\xi,\zeta_{ca}^{tr}\rangle + \langle N'_{ca}(\xi),\zeta_{ca}^{tr}\rangle + \langle\mathcal{L}_Q\xi,\zeta_{ca}^{tr}\rangle\\ \\ \langle\delta b\xi,\zeta_{ca}^{n}\rangle + \langle N'_{ca}(\xi),\zeta_{ca}^{n}\rangle + \langle\mathcal{L}_Q\xi,\zeta_{ca}^{n}\rangle\end{pmatrix},$$

and

$$B := \begin{pmatrix} \langle \xi, \zeta_{ca}^{tr} \rangle & \langle \xi, \zeta_{ca}^{n} \rangle \\ \langle \xi, \zeta_{ca}^{n} \rangle & -\langle \xi, \partial_c \partial_x^{-1} \zeta_{ca}^{n} \rangle \end{pmatrix} \Omega_{ca}^{-1}.$$

We have explicitly computed $\langle (x-a) Q_{ca}, \zeta_i \rangle$ to obtain the above expression for X.

We now estimate the error terms and solve for $\dot{a}$ and $\dot{c}$. The assumption on the potential implies the bounds

$$|\delta b| \leq \epsilon_a \epsilon_x (x-a) \text{ and } |\delta^2 b| \leq \epsilon_a \epsilon_x^2 (x-a)^2. \tag{22}$$

Thus, Hölder's inequality and exponential decay of Q_{ca} imply

$$\begin{aligned} X &= -b'(t,a)\delta(c) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathrm{O}\left(\epsilon_a \epsilon_x^2\right) \\ &= \mathrm{O}\left(\epsilon_a \epsilon_x\right). \end{aligned} \tag{23}$$

Similarly, exponential decay of ζ_{ca}^{tr} and ζ_{ca}^{n} implies $\langle \delta b \xi, \zeta_i \rangle = \mathrm{O}\left(\epsilon_a \epsilon_x \|\xi\|_{H^1}\right)$. The linear term $\langle \mathcal{L}_Q \xi, \zeta_i \rangle$ is zero since $\mathcal{L}_Q \zeta_{ca}^{tr} = 0$, $\mathcal{L}_Q \zeta_{ca}^{n} = -Q_{ca}$ and $\xi \perp \partial_x^{-1} \zeta_{ca}^{tr} = -Q_{ca}$. Lastly, $\langle N_{ca}'(\xi), \zeta_i \rangle \leq C \|\xi\|_{H^1}^2$ by the first estimate in Lemma A.3. Combining the above estimates gives the bound

$$\|Y\| = \mathrm{O}\left(\epsilon_a \epsilon_x \|\xi\|_{H^1} + \|\xi\|_{H^1}^2\right).$$

By the second inclusion of (9), $\partial_c \partial_x^{-1} \zeta_{ca}^{n} \in L^2(\mathbb{R})$. Hölder's inequality then implies $\|B\| = \mathrm{O}\left(\|\xi\|_{H^1}\right)$. Thus, if $\|\xi\|_{H^1}$ is sufficiently small, say so that $\|B\| \leq \frac{1}{2}$, then $I + B$ is invertible and $\|\left(I+B\right)^{-1}\| \leq 2$. Acting on equation (21) by $(I+B)^{-1} = I - B(I+B)^{-1}$ and then Ω_{ca}^{-1} gives the equation

$$\begin{pmatrix} \dot{a} - c + V(a) \\ \dot{c} \end{pmatrix} = \Omega_{ca}^{-1}[X + B(I-B)^{-1}X + (I-B)^{-1}Y].$$

Using the above estimates of $\|B\|$, $\|(I-B)^{-1}\|$, $\|X\|$, and $\|Y\|$ implies

$$\begin{pmatrix} \dot{a} - c + V(a) \\ \dot{c} \end{pmatrix} = \Omega_{ca}^{-1} X + \mathrm{O}\left(\epsilon_a \epsilon_x \|\xi\|_{H^1} + \|\xi\|_{H^1}^2\right).$$

Replacing X by (23) completes the proof. □

5 The Lyapunov Functional

In the last section we derived dynamical equations for the modulation parameters. These equations contain the $H^1(\mathbb{R})$ norm of the fluctuation. In this section we begin to prove a bound on ξ. Recall that the latter bound is needed to ensure that u remains close to the manifold of solitary waves M_s for long time.

We employ a Lyapunov argument with Lyapunov function

$$\Gamma_c(t) := \Lambda_{ca}(Q_{ca} + \xi) - \Lambda_{ca}(Q_{ca}) + b'(a)\langle (x-a)Q_{ca}, \xi \rangle. \tag{24}$$

Remark: if $f(u) = u^3$, the last term in the Lyapunov functional is not needed; however, apart from computational complexity, there is no disadvantage in using the above function for this special case as well.

Lemma 2. *Say $u = Q_{ca} + \xi$ is a solution to (1), where ξ satisfies (14). Say $\epsilon_a \leq 1$. If $\delta'(c) > 0$, and ϵ_x and $\|\xi\|_{H^1}$ are less than 1, with $\|\xi\|_{H^1}$ small enough, then*

$$\frac{d}{dt}\Gamma_c(t) = \mathrm{O}\left(\epsilon_a^2\epsilon_x^3 + \left(\epsilon_a\epsilon_x\epsilon_t + \epsilon_a\epsilon_x^2\right)\|\xi\|_{H^1} + \epsilon_a\epsilon_x\|\xi\|_{H^1}^2 + \|\xi\|_{H^1}^4\right). \tag{25}$$

Proof. Suppressing explicit dependence on x and t, we have by definition

$$\Lambda_{ca}(u) := H_b(u) - \frac{1}{2}\int_{-\infty}^{\infty} u^2 b\, dx + cP(u).$$

Thus, relations (4), (5) and (6) imply that the time derivative of Λ_{ca} along the solution u is

$$\frac{d}{dt}\Lambda_{ca}(u) = \int_{-\infty}^{\infty} \frac{1}{2}\dot{c}u^2 + b'\left[\frac{1}{2}cu^2 - uf(u) + \frac{3}{2}(\partial_x u)^2 + F(u)\right] + b''\, u\partial_x u\, dx.$$

Substituting $Q_{ca} + \xi$ for u, manipulating the result using antisymmetry of ∂_x, and collecting appropriate terms into $b'(a)\langle \mathcal{L}_Q\xi, \partial_x((x-a)Q_{ca})\rangle$, $\langle N'_{ca}(\xi), \partial_x[\delta b(Q_{ca}+\xi)]\rangle$, and $\langle \Lambda'_{ca}(Q_{ca}), \partial_x(\delta b(Q_{ca}+\xi))\rangle$ gives the relation

$$\begin{aligned}\frac{d}{dt}[\Lambda_{ca}(Q_{ca}+\xi) - \Lambda_{ca}(Q_{ca})] =& b'(a)\langle \mathcal{L}_Q\xi, \partial_x((x-a)Q_{ca})\rangle + \dot{c}\langle Q_{ca}, \xi\rangle + \langle \mathcal{L}_Q\xi, \partial_x\left(\delta^2 bQ_{ca}\right)\rangle + \dot{c}\frac{1}{2}\|\xi\|_{L^2}^2\\ &+ c\frac{1}{2}\langle b'\xi, \xi\rangle + \frac{3}{2}\langle b'\partial_x\xi, \partial_x\xi\rangle - \langle f'(Q_{ca})\xi, \partial_x(\delta b\xi)\rangle\\ &+ \langle N'_{ca}(\xi), \partial_x[\delta b(Q_{ca}+\xi)]\rangle + \langle b''\xi, \partial_x\xi\rangle + \langle \Lambda'_{ca}(Q_{ca}), \partial_x[\delta b(Q_{ca}+\xi)]\rangle.\end{aligned}$$

The last term is zero because $\Lambda'_{ca}(Q_{ca}) = 0$ and since $\xi \perp Q_{ca}$, the quantity $\dot{c}\langle \xi, Q_{ca}\rangle$ is also zero. We use Lemma 3, assumptions (12) on the potential, estimates (22), and

$$|\delta b'| \leq \epsilon_a\epsilon_x^2 x$$

to estimate the size of the time derivative. We also use that Q_{ca}, $\partial_x Q_{ca}$, $\partial_x^2 Q_{ca}$ and $f'(Q_{ca})$ are exponentially decaying. When $\epsilon_x \leq 1$, higher order terms like $\langle b''\xi, \partial_x\xi\rangle$ are bounded above by lower order terms like $\langle b'\xi, \xi\rangle$. Similarly, if $\|\xi\|_{H^1} \leq 1$, then $\epsilon_a\epsilon_x\|\xi\|_{H^1}^2 \leq \epsilon_a\epsilon_x\|\xi\|_{H^1}$. This procedure gives the estimate

$$\begin{aligned}\frac{d}{dt}[\Lambda_{ca}(Q_{ca}+\xi) - \Lambda_{ca}(Q_{ca})] =& b'(a)\langle \xi, \mathcal{L}_Q\partial_x((x-a)Q_{ca})\rangle + \langle N'_{ca}(\xi), \delta b\partial_x\xi\rangle\\ &+ \mathrm{O}\left(|\dot{c}|\|\xi\|_{H^1}^2 + \epsilon_a\epsilon_x^2\|\xi\|_{H^1} + \epsilon_a\epsilon_x\|\xi\|_{H^1}^2\right).\end{aligned}$$

Applying the chain rule to the integrand of

$$\int_{-\infty}^{\infty} \partial_x\left[\left(F(Q_{ca}+\xi) - F(Q_{ca}) - f(Q_{ca})\xi - \frac{1}{2}f'(Q_{ca})\xi^2\right)\delta b\right] dx = 0$$

and using the definition of $N'_{ca}(\xi)$ gives that

$$\begin{aligned}\langle N'_{ca}(\xi), \delta b\partial_x\xi\rangle =& \langle N'_{ca}(\xi) + \frac{1}{2}f''(Q_c)\xi^2, \delta b\partial_x Q_c\rangle\\ &- \int_{-\infty}^{\infty}\left(F(Q_{ca}+\xi) - F(Q_{ca}) - f(Q_{ca})\xi - \frac{1}{2}f'(Q_{ca})\xi^2\right)b'\, dx.\end{aligned}$$

The second estimate and the proof of the third estimate of Lemma 3 of Appendix A then imply the bound $\langle N'_{ca}(\xi), \delta b \partial_x \xi \rangle = \mathrm{O}\left(\epsilon_a \epsilon_x \|\xi\|_{H^1}^3\right)$. Thus, since $\epsilon_a \epsilon_x \|\xi\|_{H^1}^3 \le \epsilon_a \epsilon_x \|\xi\|_{H^1}^2$ when $\|\xi\|_{H^1} \le 1$, we have

$$\frac{d}{dt}[\Lambda_{ca}(Q_{ca}+\xi) - \Lambda_{ca}(Q_{ca})] = b'(a)\langle \xi, \mathcal{L}_Q \partial_x((x-a)Q_{ca})\rangle + \mathrm{O}\left(|\dot{c}|\|\xi\|_{H^1}^2 + \epsilon_a \epsilon_x^2 \|\xi\|_{H^1} + \epsilon_a \epsilon_x \|\xi\|_{H^1}^2\right). \tag{26}$$

When $f(u) = u^3$, $\langle \xi, \mathcal{L}_Q \partial_x((x-a)Q_{ca})\rangle = 0$ since $\zeta_{ca}^n = \partial_x[(x-a)Q_{ca}]$. In this special case the above estimate is sufficient for our purposes, but in general, we need to use the corrected Lyapunov functional. When $\xi \in C(\mathbb{R}, H^1(\mathbb{R})) \cap C^1(\mathbb{R}, H^{-2}(\mathbb{R}))$, $b'(a)\langle \xi, (x-a)Q_{ca}\rangle$ is continuously differentiable with respect to time;

$$\begin{aligned}\frac{d}{dt}\left[b'(a)\langle \xi, (x-a)Q_{ca}\rangle\right] =& \partial_t b' \langle \xi, (x-a)Q_{ca}\rangle + b'(a)\langle \dot{\xi}, (x-a)Q_{ca}\rangle + \dot{c} b'(a)\langle \xi, (x-a)\zeta_{ca}^n\rangle \\ &+ \dot{a} b'(a)\langle \xi, (x-a)\zeta_{ca}^{tr}\rangle + \dot{a} b''(a)\langle \xi, (x-a)Q_{ca}\rangle,\end{aligned}$$

where $\langle \xi, Q_{ca}\rangle = 0$ has been used to simplify the derivative. Substituting for $\partial_t \xi$ using (19) gives

$$\begin{aligned}\frac{d}{dt}[b'(a)\langle \xi, (x-a)Q_{ca}\rangle] =& - b'(a)\langle \xi, \mathcal{L}_Q \partial_x((x-a)Q_{ca})\rangle - [\dot{a} - c + b(a)]b'(a)\frac{1}{2}\|Q_{ca}\|_{L^2}^2 + \partial_t b' \langle \xi, (x-a)Q_{ca}\rangle \\ &+ [\dot{a} - c + b(a)]b'(a)\langle \partial_x \xi, (x-a)Q_{ca}\rangle + [\dot{a} - c + b(a)]b''(a)\langle \xi, (x-a)Q_{ca}\rangle \\ &+ \dot{c} b'(a)\langle \xi, (x-a)\zeta_{ca}^n\rangle - b'(a)\langle \xi, \delta b \partial_x((x-a)Q_{ca})\rangle - b'(a)\langle N'_{ca}(\xi), \partial_x((x-a)Q_{ca})\rangle \\ &- b'(a)\langle \delta^2 b Q_{ca}, \partial_x((x-a)Q_{ca})\rangle + [c - b(a)]b''(a)\langle \xi, (x-a)Q_{ca}\rangle.\end{aligned}$$

We estimate using the same assumptions used to derive (26). If $\|\xi\|_{H^1}$ and ϵ_x are less than 1, then

$$\begin{aligned}\frac{d}{dt}[b'(a)\langle \xi, (x-a)Q_{ca}\rangle] =& - b'(a)\langle \xi, \mathcal{L}_Q \partial_x((x-a)Q_{ca})\rangle + \mathrm{O}\left(|\dot{a} - c + b(a)|\epsilon_a \epsilon_x + |\dot{c}|\epsilon_a \epsilon_x \|\xi\|_{H^1}\right) \\ &+ \mathrm{O}\left(\epsilon_a^2 \epsilon_x^3 + ((1+\epsilon_a)\epsilon_x^2 + \epsilon_x \epsilon_t)\epsilon_a \|\xi\|_{H^1} + \epsilon_a \epsilon_x \|\xi\|_{H^1}^2\right).\end{aligned}$$

Adding the above expression to (26) gives an upper bound containing $|\dot{c}|$ and $|\dot{a} - c + b(a)|$. Replacing these quantities using the bounds

$$|\dot{c}| = \mathrm{O}\left(\epsilon_a \epsilon_x + \epsilon_a \epsilon_x \|\xi\|_{H^1} + \|\xi\|_{H^1}^2\right)$$

and

$$|\dot{a} - c + b(a)| = \mathrm{O}\left(\epsilon_a \epsilon_x^2 + \epsilon_a \epsilon_x \|\xi\|_{H^1} + \|\xi\|_{H^1}^2\right)$$

from Proposition 2, and bounding higher order terms by lower order terms gives (25). To use the above bounds on $|\dot{c}|$ and $|\dot{a} - c + b(a)|$ we must assume $\|\xi\|_{H^1}$ is small enough so that Proposition 2 holds. $\square$

6 Spectral Properties of the Hessian $\mathcal{L}_Q$

The Hessian $\partial^2 \Lambda_{ca}$ at Q_{ca} in the $L^2(\mathbb{R})$ pairing is computed to be the unbounded operator

$$\mathcal{L}_Q := -\partial_x^2 + c - f'(Q_{ca}), \tag{27}$$

defined on $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$. We extend this operator to the corresponding complex spaces.

Proposition 3. *The self-adjoint operator $\mathcal{L}_Q$ has the following properties.*

1. $\mathcal{L}_Q \zeta_{ca}^{tr} = 0$ *and* $\mathcal{L}_Q \zeta_{ca}^{n} = -Q_{ca}$.
2. *All eigenvalues of $\mathcal{L}_Q$ are simple, and* Null $\mathcal{L}_Q =$ Span $\{\zeta_{ca}^{tr}\}$.
3. *$\mathcal{L}_Q$ has exactly one negative eigenvalue.*
4. *The essential spectrum is $[c, \infty) \subset \mathbb{R}_+$.*
5. *$\mathcal{L}_Q$ has a finite number of eigenvalues in $(-\infty, c)$.*

Proof. Recall that the vectors $\zeta_{ca}^{tr} := -\partial_x Q_{ca}$ and $\zeta_{ca}^{n} := \partial_c Q_{ca}$ are in the Sobolev space $H^2(\mathbb{R})$. Thus, relations $\mathcal{L}_Q \zeta_{ca}^{tr} = 0$ and $\mathcal{L}_Q \zeta_{ca}^{n} = -Q_{ca}$ make sense, and are obtained by differentiating $\Lambda'_{ca}(Q_{ca}) = 0$ with respect to a and c. The first relation above proves that ζ_{ca}^{tr} is a null vector.

Say $\zeta, \eta \in H^2(\mathbb{R})$ are linearly independent eigenvectors of $\mathcal{L}_Q$ with the same eigenvalue. Then, since $\mathcal{L}_Q$ is a second order linear differential operator without a first order derivative, the Wronskian

$$W(\eta, \zeta) = \zeta \partial_x \eta - \eta \partial_x \zeta$$

is a non-zero constant. With η and ζ both in $H^2(\mathbb{R})$ however, the limit $\lim_{x\to\infty} W(\eta, \zeta)$ is zero. This contradicts the non vanishing of the Wronskian, and hence all eigenvalues of $\mathcal{L}_Q$ are simple and, in particular, Null $\mathcal{L}_Q =$ Span $\{\zeta_{ca}^{tr}\}$.

Next we prove that the operator $\mathcal{L}_Q$ has exactly one negative eigenvalue using Sturm-Liouville theory on an infinite interval. Recall that the solitary wave $Q_{ca}(x)$ is a differentiable function, symmetric about $x = a$ and monotonically decreasing if $x > a$. This implies that the null vector ζ_{ca}^{tr}, or equivalently, the derivative of Q_{ca} with respect to x, has exactly one root at $x = a$. Therefore, by Sturm-Liouville theory, zero is the second eigenvalue and there is exactly one negative eigenvalue.

We use standard methods to compute the essential spectrum. Since the function $f'(Q_{ca}(x))$ is continuous and decays to zero at infinity, the bottom of the essential spectrum begins at $\lim_{x\to\infty}(c - f'(Q_{ca}(x))) = c$ and extends to infinity: $\sigma_{ess}(\mathcal{L}_Q) = [c, \infty)$. Furthermore, the bottom of the essential spectrum is not an accumulation point of the discrete spectrum since $f'(Q_{ca}(x))$ decays faster than x^{-2} at infinity. Hence, there is at most a finite number of eigenvalues in the interval $(-\infty, c)$. For details see [33, 34, 22]. ◻

7 Strict Positivity of the Hessian

In this section we prove strict positivity of the Hessian $\mathcal{L}_Q$ on the orthogonal complement to the 2-dimensional space $\partial_x^{-1} T_{Q_{ca}} M_s =$ Span $\left\{Q_{ca}, \partial_x^{-1} \zeta_{ca}^{n}\right\}$. This result is a crucial ingredient needed to prove the bound on the fluctuation ξ.

Proposition 4. *Assume $\delta'(c) > 0$ on $I \subset \mathbb{R}_+$. If $\xi \perp \partial_x^{-1} T_{Q_{ca}} M_s$, then there is a positive constant ρ such that $\langle \mathcal{L}_Q \xi, \xi \rangle \geq \rho \|\xi\|_{H^1}^2$.*

Proof. Define $X := \{\xi \in H^1(\mathbb{R}) \mid \xi \perp \partial_x^{-1} T_{Q_{ca}} M_s,\ \|\xi\|_{L^2} = 1\}$. By the max-min principle, $\inf_{X \cap H^2(\mathbb{R})} \langle \mathcal{L}_Q \xi, \xi \rangle$ is attained or is equal to $\inf \sigma_{ess}(\mathcal{L}_Q) = c$. If the later holds the proof is complete. In the former case, let η be the minimizer.

We claim the set of vectors $\{\zeta_{ca}^{tr}, \zeta_{ca}^{n}, \eta\}$ is an linearly independent set. If they were dependent, then, since ζ_{ca}^{tr} and ζ_{ca}^{n} are orthogonal, there are non-zero constants α and β such that $\eta = \alpha\zeta_{ca}^{tr} + \beta\zeta_{ca}^{n}$. Projecting this equation onto $\partial_x^{-1}\zeta_{ca}^{tr}$ and $\partial_x^{-1}\zeta_{ca}^{n}$ gives the equations $\beta\delta'(c) = 0$ and $\alpha\delta'(c) = 0$. Thus, the assumption $\delta'(c) > 0$ implies $\eta = 0$. A contradiction since the zero function does not lie in the set X. Note that in deriving $\alpha\delta'(c) = 0$ we have used that ∂_x^{-1} is antisymmetric on the span of ζ_{ca}^{n} since $\partial_x^{-1}\zeta_{ca}^{n} \in L^2(\mathbb{R})$.

By the min-max principle, if

$$\begin{aligned}\lambda_3 &:= \inf\left\{\max\left\{\langle \mathcal{L}_Q\xi, \xi\rangle \,|\, \xi \in V,\, \|\xi\|_{L^2} = 1\right\} \,|\, V \subset H^2(\mathbb{R}),\, \dim V = 3\right\}\\ &\le \max\left\{\langle \mathcal{L}_Q\xi, \xi\rangle \,|\, \xi \in \text{Span }\left\{\zeta_{ca}^{tr},\, \zeta_{ca}^{n},\, \eta\right\}\right\}\end{aligned}$$

is below the essential spectrum, then it is the third eigenvalue counting multiplicity. Let $\xi = \alpha\eta + \beta\zeta_{ca}^{tr} + \gamma\zeta_{ca}^{n}$ where α, β and γ are arbitrary apart from satisfying $\|\xi\|_{L^2} = 1$. Thus, since the third eigenvalue of $\mathcal{L}_Q$ is positive (see Section 6),

$$0 < \langle \mathcal{L}_Q\xi, \xi\rangle = \alpha^2\langle \mathcal{L}_Q\eta, \eta\rangle - \gamma^2\delta'(c) \le \alpha^2\langle \mathcal{L}_Q\eta, \eta\rangle,$$

and hence $\langle \mathcal{L}_Q\eta, \eta\rangle > 0$. The function $\sigma(c) = \langle \mathcal{L}_Q\eta, \eta\rangle$ is continuous since both $\partial_x^{-1}\zeta_{ca}^{tr}$ and $\partial_x^{-1}\zeta_{ca}^{n}$ are continuous in $L^2(\mathbb{R})$ as functions of c. Set $\varrho = \inf_I \sigma(c)$.

We now improve the result to an $H^1(\mathbb{R})$ norm. If we define the constant $K(I) := \sup_I \|c - f'(Q_{ca})\|_\infty$, then $\langle \mathcal{L}_Q\xi, \xi\rangle \ge \|\partial_x\xi\|_{L^2}^2 - K(I)\|\xi\|_{L^2}^2$. Adding to this bound the factor $\frac{K+1}{\varrho}$ of the lower bound $\langle \mathcal{L}_Q\xi, \xi\rangle \ge \varrho\|\xi\|_{L^2}^2$ derived above completes the proof. $\square$

8 Bound on the Fluctuation

We are now ready to prove the bound on the fluctuation.

Proposition 5. *Say $\epsilon_a \le 1$. Then, for small enough $\epsilon_x \le 1$ and initial fluctuation $\|\xi(0)\|_{H^1} \le 1$, there exists a constant C such that the bound*

$$\|\xi(t)\|_{H^1} = \mathrm{O}\left(\epsilon_0 + (\epsilon_a\epsilon_x)^{\frac{1}{2}}\,\epsilon_0^{\frac{1}{2}} + \epsilon_x + \epsilon_t\right)$$

holds for all times $t \le T = C\,(\epsilon_a\epsilon_x)^{-1}$.

Proof. Lemma 2 implies

$$\left|\frac{d}{dt}\Gamma_c(t)\right| \le C\left(\epsilon_a^2\epsilon_x^3 + \left(\epsilon_a\epsilon_x\epsilon_t + \epsilon_a\epsilon_x^2\right)\|\xi\|_T + \epsilon_a\epsilon_x\|\xi\|_T^2 + \|\xi\|_T^4\right)$$

for some constant $C > 0$ where $\|\xi\|_T := \sup_{0\le t\le T}\|\xi\|_{H^1}$. Integrating over $[0, T]$ gives an upper bound on $\Gamma_c(T)$. A lower bound is obtained by expanding $\Lambda_{ca}(Q_{ca} + \xi)$ to quadratic order then using Proposition 4, the third estimate of Lemma 3 and $V'(a)\langle \xi, (x-a)Q_{ca}\rangle = \mathrm{O}\left(\epsilon_a\epsilon_x\|\xi\|_{H^1}\right)$. We obtain, after setting all non-essential constants to one,

$$\|\xi\|_T^2 - \|\xi\|_T^3 - \epsilon_a\epsilon_x\|\xi\|_T \le \Gamma_c(T) \le |\Gamma_c(0)| + \left(\epsilon_a^2\epsilon_x^3 + \left(\epsilon_a\epsilon_x\epsilon_t + \epsilon_a\epsilon_x^2\right)\|\xi\|_T + \epsilon_a\epsilon_x\|\xi\|_T^2 + \|\xi\|_T^4\right)T$$

for all $T > 0$. Take $T = \mathrm{O}\left((\epsilon_a\epsilon_x)^{-1}\right)$. Then, under the smallness assumption $\|\xi\|_{H^1} \ll (\epsilon_a\epsilon_x)^{\frac{1}{2}}$,

$$\|\xi\|_{H^1} = \mathrm{O}\left(|\Gamma_c(0)|^{\frac{1}{2}} + \epsilon_x + \epsilon_t\right).$$

The initial value of the Lyapunov functional $\Gamma_c(0)$ can be bounded by the $H^1(\mathbb{R})$ norm of the initial fluctuation $\|\xi(0)\|_{H^1} \leq C\epsilon_0$ (recall that $\epsilon_0 := \inf_{Q_{ca}\in M_s} \|u_0 - Q_{ca}\|_{H^1}$. Indeed, Taylor expanding $\Lambda_{ca}(Q_{ca}+\xi)$ to second order in ξ and using the third estimate in Lemma 3 gives $|\Gamma_c(0)| = \mathrm{O}\left(\epsilon_0^2 + \epsilon_a\epsilon_x\epsilon_0\right)$ if $\epsilon_0 \ll 1$. To complete the proof we take ϵ_x and ϵ_0 small enough so that $\|\xi(t)\|_{H^1}$ is sufficiently small for Lemma 2 to hold. □

We now prove the main theorem.

Proof of Theorem 1. By our choice $\epsilon_0 < \varepsilon_0$, there is a (maximal) time T_0 such that the solution u in (1) is in U_{ε_0} for time $t \leq T_0$. Hence decomposition (13) with (14), and Proposition 5 are valid for the solution u over this time and imply the statements of the main theorem. In particular $\|\xi(t)\|_{H^1} = \mathrm{O}\left(\epsilon_0 + (\epsilon_a\epsilon_x)^{\frac{1}{2}}\epsilon_0^{\frac{1}{2}} + \epsilon_x + \epsilon_t\right)$ for times $t \leq \min\{T_0, T\}$. Taking $\epsilon_0 + (\epsilon_a\epsilon_x)^{\frac{1}{2}}\epsilon_0^{\frac{1}{2}} + \epsilon_x + \epsilon_t \ll \varepsilon_0$, we must have $t \leq T$ by maximality of the time T_0. □

A Estimates of Nonlinear Remainders

Define

$$N_{ca}(\xi) := -\int_{-\infty}^{\infty} F(Q_c + \xi) - F(Q_c) - F'(Q_c)\xi - \frac{1}{2}F''(Q_c)\xi^2\, dx$$

and

$$N'_{ca}(\xi) := -\left(f(Q_c + \xi) - f(Q_c) - f'(Q_c)\xi\right). \tag{28}$$

Note that $N'_{ca}(\xi) = \partial_\xi N_{ca}(\xi)$ under the $L^2(\mathbb{R})$ pairing.

Lemma 3. *If $\|\xi\|_{H^1} \leq 1$ and $f \in C^k(\mathbb{R})$ for some $k \geq 3$, with $f^{(k)} \in L^\infty(\mathbb{R})$, then there are positive constant C_1, C_2, and C_3 such that*

1. $\|N'_{ca}(\xi)\|_{L^2} \leq C_1\|\xi\|_{H^1}^2$
2. $\|N'_{ca}(\xi) + \frac{1}{2}f''(Q_{ca})\xi^2\|_{L^2} \leq C_2\|\xi\|_{H^1}^3$
3. $|N_{ca}(\xi)| \leq C_3\|\xi\|_{H^1}^3$.

Proof. Taylor's remainder theorem implies

$$N'_{ca}(\xi) = -\sum_{n=2}^{k-1}\frac{1}{n!}f^{(n)}(Q_{ca})\xi^n - R(Q_{ca}, \xi),$$

where, since $f^{(k)} \in L^\infty(\mathbb{R})$, $|R(Q_{ca}, \xi)| \leq C|\xi|^k$. Recall that Q_{ca} is continuous and decays exponentially to zero. Together with the assumption that $f \in C^k(\mathbb{R})$, this implies $f^{(n)}(Q_{ca}) \in L^\infty(\mathbb{R})$ for $2 \leq n \leq k-1$. Thus, after pulling out the largest constant,

$$\|N'_{ca}(\xi)\|_{L^2} \leq C \sum_{n=2}^{k} \|\xi^n\|_{L^2}.$$

To obtain item 1 we use the bound $\|\xi^n\|_{L^2} \leq C\|\xi\|^n_{H^1}$, which is obtained from the inequality $\|\xi\|_{L^\infty} \leq C\|\xi\|_{H^1}$ and the assumption that $\|\xi\|_{H^1} \leq 1$.

Clearly, slight modification of the above proof gives items 2 and 3. For the latter we use that the assumptions on f imply $F \in C^{k+1}(\mathbb{R})$ with $F^{(k+1)} \in L^\infty(\mathbb{R})$. $\square$

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